Sizhe Liu

Notes on Heat Transfer Labs

University of Illinois at Urbana-Champaign, April 2021

Contents

List of Figures

6

List of Tables

To those who tried to make ME320 lab better

Preface

"I want to know something about everything after I start my career as a system engineer" said Caitlyn when we were chatting while doing a convection experiment in ME320 heat transfer lab. Her words somehow tug at my heartstrings as that was one of my goals when I started my journey as a naive Ph.D. student. After 5 years into my Ph.D. program, I am still not confident that I know enough to know everything. My failure of growing into a knowledgeable researcher might be attributed to my consistent tendency of being attracted by random things. I still waste half of my day sometimes skimming through previews of science/math books on amazon just because I found the titles suit my mood.

It was my job as a lab TA that forced me to unite all the random things in my mind, and recast them into complete stories. My original plan was just to make one or two notes for students who are not able to attend my sessions due to their exposure to COVID. In order to keep myself focused during the writing process, some impractical thought experiments are inevitably added in the first two notes. Unexpectedly, I found myself learning new things organically by developing "stories" in the notes. Things that were never mentioned in my undergrad/Ph.D. classes suddenly become indispensable to deliver vivid explanations of physical phenomena. That was the moment when I decided to finish the notes for all the labs, and see what else I can bring to the table where people are bored of learning things off textbooks.

The notes shown here defines the best part of me both as a learner and as a researcher. Writing them is a journey of self-redemption during which I proved, at least to myself, that I might be able to derive and create cool things by using simple mathematical language and intuitions of my own. I want to give my special thinks to some of my committed students: Rena Kanegae, Caitlyn Peters, Stefan Kamzol, Ashish Mittal, Haley Middendorf, and Nathan Asad for their thought-provoking questions and suggestions to the labs. And I will never forget the kind comments from Dr. Hongliang Qian and my buddy Arjun Sanjay Goswami.

–Sizhe Liu,
UIUC

Lab1 *Temperature Measurements*

> ```
> Any knowledge that doesn't lead to new questions quickly
> dies out:  it fails to maintain the temperature required
> for sustaining life.---Wislawa Szymborska
> ```

This note is mainly about how we understand the concept of "*temperature*" in modern scientific discussion, and the tools for measuring temperature. For practical-oriented minds, the second section lists advantages and disadvantages of different tools, and the scenarios that would best fit the uses of various tools.

1.1 *Temperature as a Measurement of Kinetic Energy*

If you ask a random student of physical science disciplines what *is* temperature exactly, the answer would likely be "*indicator of average kinetic energy of particles*". This argument was first implied by Swiss Physicist Danial Bernoulli in his 1738 paper [1]. But it was not widely accepted during 18th century as scientists at that time believed that heat is kind of "*fluid*" called "*caloric*", rather than the energy of atomic motions [2]. Several physicists had been worked on the idea of treating heat as a form of atomic motion since Bernoulli, but most of their results passed unnoticed until another scientific giant, James Maxwell, formulated his dynamical theory of gases in 1867 [3]. To unpack this seemingly strange argument, we start from phenomena at atomic level and later scale them up to macroscopic scale. For gaseous systems, the relationship that bridges these two vastly different scales is the well-known "*ideal gas law*",

$$PV = nRT, \qquad (1.1)$$

[1] Daniel Bernoulli. Hydrodynamica. *Dulsecker. Consultable en ligne http://imgbase-scd-ulp. u-strasbg. fr/displayimage. php*, 1738

[2] Stephen G Brush. History of the kinetic theory of gases. *Istituto della Enciclopedia Italiana*, 1, 2004

[3] JC Maxwell. Illustrations of the dynamical theory of gases. *Philos Mag*, 19:19–32, 1867

with all the symbols taking their common meanings. Because what we will deal with in this lab is mostly the ambient environment, here we derive a relationship between kinetic energy of gas molecules and temperature to prove the *energy-temperature argument*. Our strategy is simple. Because temperature is already defined at the RHS of Eq.(1.1), we will try to relate PV to energy of air molecules. Notice that *pressure $\times$ volume* has an unit of energy, so our intuition tells us that *energy of air molecule might be related to pressure*. To verify our intuition, we first make the following assumptions in a container full of gas molecules:

I.   gas molecules are treated as point mass in the container,

II.   potential energy of each molecule is negligible,

III.   the collisions between two molecules and between molecules and container walls are elastic, and

IV.   the wall of container is perfectly smooth and has no frictional effect.

The first assumption ignores inner degrees of freedom(DOFs) of gas molecules [4]. The validity of the assumption lies in the fact that the dimension of molecule is small compared with the size of container. The assumption II implies the container is small enough so gravitational potential energy does not vary dramatically. The last two assumptions, however, deserve a closer look as they dictate the outcome of molecular collisions.

[4] For polyatomic molecules, inner DOFs include the rotational and vibrational movements

1.1.1   *Collisions at the wall: momentum exchange*

There are two kinds of collisions: molecule-molecule collisions and molecule-wall collision. We now discuss them one-by-one by first setting molecular weight to be m for all the air molecules. After a collision between two molecules, their velocities swap due to the assumption III. When a molecule collides with the wall of container, because of assumption IV, its tangential velocity $\vec{v}_t$ remains

unchanged while its normal velocity $\vec{v}_n$ is flipped to its opposite equivalent (see Fig.1.1).

The change in molecular momentum p during the collision in Fig.1.1 is simply $\Delta p = 2m\vec{v}_n = 2m\vec{v}\cos(\theta)$, with θ the angle between incoming velocity and its normal component. According to Newton's second law, *the rate of momentum change equals to the force that single air molecule acts on the wall of container.* We also know that the pressure is the total force that gas molecules act on the wall per unit area, so we can relate pressure P with Δp through the relationship

Figure 1.1: Molecular velocity before and after a collision between air molecule (the circle) and the wall of container

$$P = \frac{1}{S}\,\Sigma_{N_{tot}}\frac{dp_i}{dt}\bigg|_\tau \overset{\text{Newton's Law}}{=} \frac{1}{S}\Sigma_{N_{tot}}F_{i,\tau}, \tag{1.2}$$

where N_{tot} is the total number of molecules colliding with the wall at time instance τ, and S is the inner surface area of the container. $\frac{dp_i}{dt}|_\tau$ and $F_{i,\tau}$ are the *"rate of change in momentum and molecular force of **ith** molecule at the time instant τ, respectively"*. The relation in Eq.(1.2) is coherent with the strategy of relating "pressure" to molecular properties, but it faces a big problem because of the assumption III. The elastic collision demands the action of gas molecule to be instantaneous at the wall, resulting a divergent definition of $\frac{dp_i}{dt}|_\tau$. Based on our discussion of Fig.1.1, the time derivative in Eq.(1.2) is, unfortunately,

$$\frac{dp_i}{dt}\bigg|_\tau = \frac{m\vec{v}' - m\vec{v}}{0} = \infty. \tag{1.3}$$

Instead of modifying our previous assumptions, we can add one more assumption to workaround this awkward situation, that is,

V. the directions of molecular velocities have an uniform distribution.

The assumption V derives from the fact that our gaseous system is homogeneous and still, so there is no reason to suspect that a large portion of gas molecules move at the same direction. As we will see later, assumption V allows us to replace the troublesome derivative with averaged physical quantities, from which a "pressure-energy" relationship naturally emerges. But right now, we are running out

of ammunition as the math we developed so far is meant to describe actions of single molecule not "averaged" molecular action. Thus, we need a new way to describe gas molecules.

Let's now imagine that each gas molecule has a velocity vector attached to it. We collect all of the vectors and attach their tails to the origin of a spherical coordinate system. By doing so, we can assign a 3-tuple coordinate to each vector using its magnitude v, polar angle θ, and azimuthal angle ϕ (see Fig.1.2). For a sphere of radius $|\vec{v_1}|$, it contains all the velocities that have magnitudes less or equal to $|\vec{v_1}|$. *By counting the number of intersection points between the sphere and velocity vectors, we know how many gas molecules that move with velocities no slower than $\vec{v_1}$.* On the other hand, if we draw another sphere of radius $|\vec{v_2}| > |\vec{v_1}|$ in Fig. 1.2, then the absolute difference in the intersection numbers on the two spheres tells us *the number of gas numbers that move with velocities in a range from $\vec{v_1}$ to $\vec{v_2}$.*

We can call gas molecule by the spherical coordinates of its velocity for simplicity, i.e., for molecules moving with a velocity of (θ, ϕ, v), we name them as $v\theta\phi-$molecules. With this terminology, we are well-equipped to tackle the infinity derivative in Eq.(1.3) by defining averaged quantities based on another concept: mass flux.
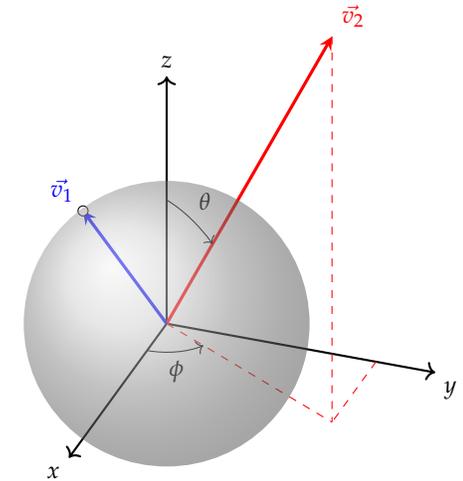
Figure 1.2: Spherical coordinate system for velocity vectors where the shaded sphere contains vectors with magnitudes less or equal to $|\vec{v_1}|$.

1.1.2   Collisions at the wall: mass flux

The awkward infinity derivative in Eq.(1.3) would go away if we focus on what is happening on the wall within a finite amount of time rather than any specific time instance. But wait, there is a lot happening during a finite time period $\Delta\tau$, how can we track movements of all the gas molecules in the container? The answer is we can't and we don't need to. Since the pressure is measured at the wall of container, we only need to care about molecules that arrive at the wall within $\Delta\tau$. *Given arbitrary area dS on the wall of container, $v\theta\phi-$molecules that will arrive dS are contained in a slant cylinder of side length $v\Delta\tau$ (see Fig.1.3)* [5]. If we let $N(v, \theta, \phi)$ be the number of

[5] It's easy to prove this argument. For molecules that move at directions very different from $\vec{v}$ near the wall, they will landed on the area outside of dS after $\Delta\tau$. One the other hand, molecules moving in the same velocity $\vec{v}$ but outside of the cylinder cannot reach dS within $\Delta\tau$.

$v\theta\phi-$molecules, then the number of molecules in the slant cylinders $N_s(v,\theta,\phi)$ can be calculated as

$$N_s(v,\theta,\phi) = \rho_{v\theta\phi} v \Delta\tau \cos(\theta) dS \tag{1.4}$$

where $\rho_{v\theta\phi}$ is the **number density** of $v\theta\phi-$molecules. To get an expression of $\rho_{v\theta\phi}$, we use again the spherical coordinate system in Fig. 1.2. Let N_v be the number of intersection points (i.e. number of molecules) between a sphere of radius v and velocity vectors that have magnitudes no less than v. Because of the assumption V, the average number of intersections per unit spherical area is $N_v/4\pi v^2$. From this, the number of molecules moving in a direction between θ and $\theta + \Delta\theta$, and between ϕ and $\phi + \Delta\phi$ is

$$N_{\theta\phi} = \frac{N_v}{4\pi v^2} v^2 \sin(\theta)\Delta\theta\Delta\phi = \frac{N}{4\pi} \sin(\theta)\Delta\theta\Delta\phi. \tag{1.5}$$

From Eq.(1.5), we can further obtain the number of $\theta\phi-$molecules that have their velocities between v amd $v + \Delta v$ to be

$$n_{v\theta\phi} = \frac{N_v - N_{v+\Delta v}}{4\pi} \sin(\theta)\Delta\theta\Delta\phi = \frac{\Delta N_v}{4\pi} \sin(\theta)\Delta\theta\Delta\phi, \tag{1.6}$$

and

$$\rho_{v\theta\phi} = \frac{n_{v\theta\phi}}{V} = \frac{\Delta N_v}{4\pi V} \sin(\theta)\Delta\theta\Delta\phi \tag{1.7}$$

where V is the volume of container.

Dividing both sides of Eq. (1.4) by $\Delta\tau dS$ gives us the **number of $v\theta\phi-$molecules that collide with the wall per unit time per unit wall area**, $\Phi(v,\theta,\phi)$ [6], i.e.,

$$\Phi(v,\theta,\phi) = \rho_{v\theta\phi} v \cos(\theta). \tag{1.8}$$

$\Phi(v,\theta,\phi)$ is referred as mass flux of $v\theta\phi-$molecules.

Now that we have well-defined mass fluxes that is "time-derivative", we can free p_i in Eq.(1.2) from the awkward $\frac{d}{dt}$, and applied the derivative to its prefactor "$\Sigma_{N_{tot}}/S$". To do so, we need to recast $\Sigma_{N_{tot}}/S$ into a form with mass fluxes in it.
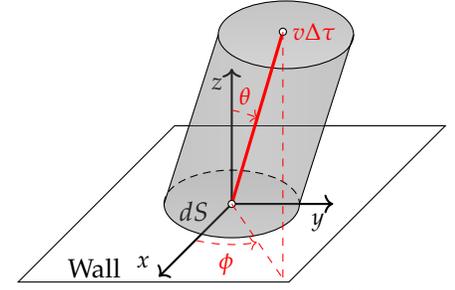
Figure 1.3: $v\theta\phi-$molecules that will arrive at S within next $\Delta\tau$ period must be found in a slant cylinder with side length being $v\Delta\tau$.

[6] Notice that $\Phi(v,\theta,\phi)$ has an unit of *number/time/area*. It essentially tells number of molecules that arrive at an unit of area per unit time. And we now have a well-defined "time-derivative" quantity to play with!

1.1.3   *A ride back to ideal gas law*

To recast Eq.(1.2), we go back to Fig.1.1 and notice that the change in momentum of $v\theta\phi-$molecule after a collision is still

$$\Delta p = 2mv\cos(\theta), \tag{1.9}$$

which is again independent of azimuthal angle ϕ. The character of *azimuthal angle free* of the container has something to do with the assumption IV. Imagine there are receptors on the wall that absorb molecules only when they move in specific polar and azimuthal angles, then Δp will be $\phi-$dependent [7]. But we will just stick to Eq.(1.9) without losing too much generality. Now, the trick is: if we multiply Δp with $\Phi(v,\theta,\phi)$, the product has an unit of [momentum $\times$ molecule number/(time $\times$ area)], identical to that of pressure! Thus, we conclude that *the product $\Delta p \times \Phi(v,\theta,\phi)$ gives the pressure that is caused by bombarding of the $v\theta-$molecules, and is felt by the wall of container.* The pressure is therefore the integration of $\Delta p \times \Phi(v,\theta,\phi)$ over θ, ϕ, and v. Combining Eq.(1.7), (1.8) and (1.9) gives

[7] Physicists call this independent-dependent transition as "break in symmetry", and theory that studies symmetries of physical problems is called *group theory*.

$$P = \Sigma_v \int_0^{2\pi} \int_0^{\pi/2} \frac{\Delta N_v}{2\pi V} mv^2 \sin(\theta)\cos^2(\theta)d\theta d\phi = \frac{m}{3}\Sigma_v \frac{\Delta N_v v^2}{V}, \tag{1.10}$$

where Σ_v is a summation over all possible magnitudes of velocity. The last equivalence in Eq.(1.10) hides the average of v^2, i.e. [8],

[8] N_{tot} here is still the total number of gas molecules in the container

$$\bar{v^2} = \frac{\Sigma_v \Delta N_v v^2}{N_{tot}}. \tag{1.11}$$

So we can write

$$PV = N_{tot}\frac{1}{3}m\bar{v^2} = nRT. \tag{1.12}$$

The middle part in the equation above is very much like averaged kinetic energy of particles times total number of molecules. If we write the gas constant R as a product of Avogadro's constant N_A and Boltzmann constant k_B, we have $N_{tot} = nN_A$ and finally,

$$\frac{1}{2}m\bar{v^2} = \frac{3k_B T}{2}. \tag{1.13}$$

Eq.(1.13) clearly shows that temperature relates to the averaged kinetic energy of gas molecule by a constant $\frac{3k_B}{2}$. Because the container has a 3D space, and we have the velocity squared to be $v^2 = v_x^2 + v_y^2 + v_z^2$. Eq.(1.13) also indicates that the averaged kinetic energy is simply $k_B T/2$ at each dimension. □

1.2 *Tools for Measuring Temperature*

Tools for measuring temperature can be roughly categorized into three classes: thermoelectric, electro-activated, and radiation-activated. In this section, we will emphasize the use of the first two classes, and their primary differences are listed in Table.1.1.

	Thermoelectric	Electro-activated	Radiation-activated
Principle	Seebeck effect	temperature-resistance relation	radiative heat transfer
Accuracy	intermediate	high/intermediate	intermediate/low
Application	point-wise measurements	point-wise mesurements	area-averaged measurements

Table 1.1: Comparison between different classes of temperature measurement tools

Thermoelectric tools rely on Seebeck effect where temperature gradient in thermally conductive material induces electron flow and establish finite electric potential drop from hot end to cold end. The possibly simplest application of Seebrck effect is thermocouple in which two wires made of dissimilar materials join at one end and connect to voltage meter at the other end. When the joint is heating up or cooling down, two distinct potential drops establish in two wires, and their difference is measured at the voltage meter. As what Fig.1.4 shows, if two wires made of identical material are joined, no net voltage will be established at voltage meter, and no meaningful temperature reading will be obtained.

Unlike thermoelectric tools, electro-activated tools require activation of current in inner circuit where resistance of alloy material is measured. It is well-known that electric resistance of metallic materials varies with temperature. Materials that have their resistances
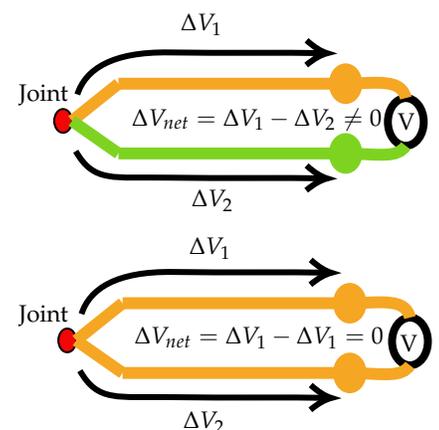
Figure 1.4: Seebeck effect in thermocouple where two dissimilar materials establish non-zero potential drop at voltage meter (top) while two identical materials establish zero potential drop (bottom)

increasing/decreasing with increasing temperature are said to have positive/negative temperature coefficients(or PTC/NTC). Among all the electro-activated tools, resistance temperature detector(RTD) and thermistor are the most popular ones. RTD usually uses materials with PTC, which have nearly linear relationship between temperature and resistance [9]. As a result, *measurements from RTD are repeatable and thus more reliable than thermistor*. However, the typical activation current in RTD is 1 mA or less, resulting in much longer waiting time before the device reaches its steady state. Thermistor usually uses NTC material, which has steep resistance-temperature curve at low temperature range (see Fig.1.5). Because of this, thermistor is very sensitive when measuring low/intermediate temperatures, but the nonlinearity of resistance-temperature curve could cause inaccuracy in measurements.

Finally, radiation-activated tools use bolometer array to receive electromagnetic radiation from the surface of target. The energy carried by electromagnetic radiation is transfomed into thermal energy to heat up thermometer in inner circuit to give temperature readings. As of Aprial,2021, a large amount of infrared thermometers have been employed in the COVID testing process, and they can also be grouped into the class of radiation-activated tools.

[9] See some of common temperature-resistance curves here
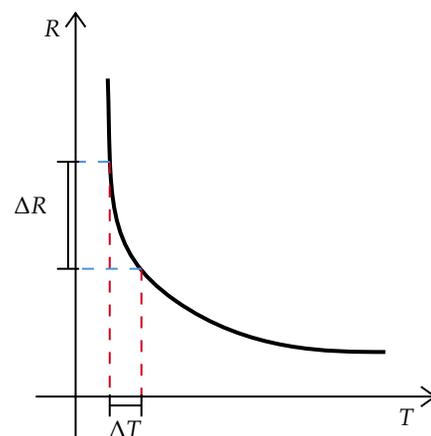
Figure 1.5: Resistance-temperature curve for NTC material, where steep gradient is found at low temperature

1.3   *Few More Words on Infinite Time Derivative*

The infinite time derivative introduced in Eq.(1.3) might puzzle readers who have previous experience of experimenting classical-mechanical phenomena, such as potential-kinetic energy transformation. Everything people observe in those experiments is smooth and differentiable. The idea of infinite time derivative there is, simply unfeasible. The most famous counter example is perhaps Brownian motion, which describes random behavior of small particles. By taking a closer look at the trajectories of Brownian motion, Einstein, quite shockingly, found that it is not possible to find well-

defined derivative anywhere along the trajectory. Such bizarre fea-
ture of Brownian motion, as shown by Feynman [10], can be related to
the stochastic nature of our world.

[10] Richard P Feynman, Albert R
Hibbs, and Daniel F Styer. *Quantum
mechanics and path integrals*. Courier
Corporation, 2010

Lab2 *Basics of Heat Conduction, Inductively*

In this note, we summarize basics of heat conduction using inductive logic. By doing so we may be able to internalize important things by building conceptual connections among them. Hopefully, Delivering the content inductively can make one feels that we are building a tunnel together from nowhere to a place where things can be understood in terms of temperature gradient, Fourier's law, and heat flux.

2.1 *Two concepts: Thermal energy and Specific Heat*

We first deduce a relationship between thermal energy and specific heat. In a heat transfer analysis we essentially deal with thermal energy flow from hot regions to cold regions. Such energy transfer process is manifested by time-(in)dependent temperature distributions. Inspired by these facts, we conclude that there must be a relationship between heat energy and temperature. Like many other physical problems, we don't have an "absolute reference" to define the zero point of thermal energy. Instead, we only care about the differences in thermal energy between two objects/regions. So it is most likely that difference in thermal energy is directly related to temperature difference. In other words, the thermal energy difference ΔQ, is a function of temperature difference ΔT, i.e.,

$$\Delta Q = f(\Delta T) \tag{2.1}$$

To figure out the exact form of the function in Eq.(2.1), we need help from Joseph Black, a scottish experimentalist who found that

```
"equal masses of different substances require different
among of heating time to reach a preset target
temperature."
```

Let's assume that the heater Joseph used had a steady power output, P, then, the thermal energy injected into each substance in Joseph's experiment can be calculated as the product of P and heating time t, i.e., $\Delta Q = P \times t$. By plotting ΔQ versus ΔT of each substance, Joseph found a linear relation which says

$$\Delta Q = \alpha \Delta T, \tag{2.2}$$

with α being a constant, different for each substance. Because two objects made of identical substance but differring in masses require different amount of thermal energy to achieve a given temperature, we know α varies with mass m. If we fix ΔT in Eq.(2.2) and vary the sample mass in Joseph's experiment, we can further plot ΔQ versus m to find that

$$\Delta Q = mc_p\Delta T, \tag{2.3}$$

where c_p is a coefficient called **specific heat**, and has an unit of energy per mass. In 1819, French chemist Pierre-Louis Dulong and the French physicist Alexis-Thérèse Petit found that

```
"atomic weight times specific heat of an element is a
constant", i.e., Dulong-Petit law.
```

Since then, people use such law to measure atomic weights for different elements. During the first decade of 20th century, Dulong-Petit law was found broken as specific heat of any substance exponentially decreases when environmental temperature drops to a extremely low value. The deviation from Dulong-Petit Law inspired Einstein and Debye to understand the root of specific heat in terms of **lattice vibrations**. It was Debye who first treated lattice vibrations as bosonic particles, called **phonons** [1]. Debye's model, along with Planck's law of black body radiation, marked the beginning of quantum mechanics.

[1] Debye's model on Wikipedia

Obviously, Eq.(2.3) is not handy to use as we are not excited to know the mass of the substance on which we observe heat transfer phenomena. To make Eq.(2.3) more general, we normalize both sides of the equation by a characteristic volume to give

$$\Delta Q = \rho c_p \Delta T. \tag{2.4}$$

We now have the mass density ρ in Eq.(2.4), and we will use this equation later to derive heat equation in 1D media.

2.2   *Fourier's Law: Governing Equation for Heat Transport in Materials*

As what alluded above, the capacity of a material storing thermal energy can be attributed to lattice vibrations. However, **it is electrons that transport thermal energy within a material**. People know this because of the Seebeck effect where a potential difference is built between two ends of a wire because of heat transport. From this point of view, we need a new equation to describe the rate of heat transport. To derive such a equation, we first define the concept of "**heat flux**",q, as **flow of thermal energy per unit area per unit time. It has a unit of watts/area**. In his experimental paper published in 1822, Fourier concluded that

```
"the heat flux resulting from thermal conduction is
proportional to the magnitude of the temperature gradient
and opposite to it in sign."
```

From this conclustion, Fourier's law can be written as:

$$q = -\kappa \nabla T, \tag{2.5}$$

where κ is a coefficient called **thermal conductivity**. Eq.(2.5), along with the concept of thermal flux, sets up a stage for us to make an **analogy between heat conduction and electric current**. In an electric circuit, we have a relationship among resistance R, voltage V, current density i, and cross-section area A, as

$$\Delta V = i \times A \times R. \tag{2.6}$$

The current density i is defined as the amount of electrons passing through per unit area per unit time. Now, if we imagine the heat conduction as a process of energy flowing through a material, we can replace current i in (2.6) with heat flux q, and replace potential difference ΔV with ΔT to give,

$$\Delta T = q \times A \times R_t. \tag{2.7}$$

Eq.(2.7) gives the definition of "**thermal resistance**". For an isotropic 1D material of length L, if temperatures at two ends are T_2 and T_1, Eq.(2.5) can be written as

$$q = -\kappa \nabla T = -\kappa \frac{T_2 - T_1}{L}, \tag{2.8}$$

under the steady-state condition. Substituting (2.8) into (2.7) gives the explicit expression for R_t as

$$R_t = \frac{(T_1 - T_2)}{-\kappa(T_2 - T_1)A/L} = \frac{L}{\kappa A}. \tag{2.9}$$

With this current-heat flux analogy, we can easily apply rules of serial/parallel resistance in electric circuit to our thermal circuit.

2.2.1   *Contact thermal resistance*

We can treat heat conduction at an interface between two bodies as heat flow through a contact thermal resistance. Let lengths of body A and B be L_A and L_B, and their cross sections be A, so their thermal resistances are $R_{t,A} = \frac{L}{\kappa_A A}$ and $R_{t,B} = \frac{L_B}{\kappa_B A}$. At the interface between A and B, we don't have the concept of thermal conductivity well defined, instead, we define the **thermal conductance coefficient**, h_c, by writting the contact thermal resistance $R_{t,c}$ as $\frac{1}{h_c A}$. Therefore, the heat flow between the two bodies in contact, bodies A and B, is found as

$$q = \frac{T_1 - T_2}{\Delta x_A / (k_A A) + 1/ (h_c A) + \Delta x_B / (k_B A)},$$ (2.10)

where T_1 and T_2 are temperatures at two ends.

2.3 *Heat Equation: Thermal Energy, Specific Heat, and Fourier's Law in one Place*

Let's imagine an 1D rod of cross section area A. If we only focus on a thin slice of the rod with a thickness of dx, we know, from Eq.(2.3) and (2.4), that the change rate of thermal energy in the slice is

$$\frac{\partial Q}{\partial t} = \rho c_p A(dx) \frac{\partial T}{\partial t}.$$ (2.11)

On the other hand, the heat flux at the right and left faces of the slice are q_l and q_r. By applying the conservation law of energy, we have

$$\frac{\partial Q}{\partial t} = (q_l - q_r)A.$$ (2.12)

Equating the RHS of equations (2.11) and (2.12) gives:

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}.$$ (2.13)

Using Fourier's law at RHS of (2.13), we finally arrive at the 1D **heat equation**:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_p} \frac{\partial}{\partial x} \left(\kappa \frac{\partial T}{\partial x} \right).$$ (2.14)

In 3D space, Eq.(2.14) is generalized as:

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right).$$ (2.15)

where $\alpha = \frac{\kappa}{\rho c_p}$ is the thermal diffusivity. Here we assume thermal conductivity κ to be constant. To investigate temperature distri-

bution on a disk, we need to write Eq.(2.15) in a cylindrical coordinate as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot k\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho c_p \frac{\partial T}{\partial t}. \qquad (2.16)$$

2.4 *Summary*

Things are not always derived from fist principles. Fourier's law has no rigorous root in mathematical physics, even though people have tried to derive it from basic axioms of quantum mechanics [2]. But as long as people are satisfied with the concept of energy flow and heat flux, we as human being should have no problem of using it. That is, we use things before we know what they really are. Looks like that's just basics of how to build mordern civilizations.

[2] see here

Lab3 Basics of Convective and Radiative Heat Transfer

This chapter explains basic aspects of convection and radiation, the two primary ways of heat transfer other than conduction. We wish to understand the concepts of buoyancy, forced convection, radiation from first principles. The hope is to establish the importance of different boundary conditions in heat

3.1 Newton's Cooling Law: When the Heat Conductor is Exposed

3.1.1 When the insulator is still there

In Lab1 we discuss the heat transfer process only by conduction, where we force thermal energy flowing at radial and axial directions by cutting off heat exchange between the conductor and ambient environment. Let's first review the setup of linear heat conduction in Lab2 (see Fig.3.1), where we have a metal rod isolated from surrounding environment by thermal insulator. We already discussed that the temperature profile at axial direction is linear as the (1D) heat equation

Figure 3.1: Schematic of one-dimensional conduction.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

at steady state becomes

$$0 = \alpha \frac{\partial^2 T}{\partial x^2},$$

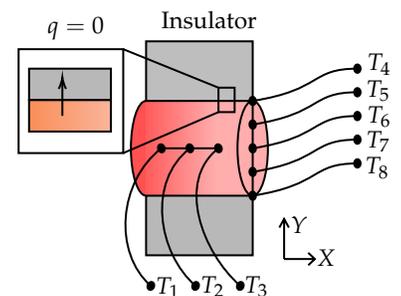which has a simple solution of $T = ax + b$. But *what if we want to*

be more realistic by considering the heat transfer along the radial direction?

We can develop our intuition of radial temperature profile by first considering boundary condition at the radial direction. Because the surface of our metal rod has a direct contact with the thermal insulator, it is forbidden for any heat transferred at the interface between the rod and the insulator. This brings up the boundary condition of zero heat flux, i.e., $q = 0$. Remember that **Fourier's law** tells us that the driving force of heat flow is temperature difference, so $q = 0$ indicates **no temperature difference at the interface**. As a result, the temperature profile near the interface must be flat, or, **the temperature gradient must vanish at the interface according to Fourier's law**. Now, imagine that **the system in Fig.3.1 haven't reached its steady state**, and that the insulator is colder than the rod at the beginning. By enforcing the boundary condition of $q = 0$, the axial temperature profile should look like the solid line in the bottom subplot of Fig.3.2. As time passes by, the hot center of the rod dissipates heat radially to the rod-insulator interface. Again, by enforcing the boundary condition, the profile becomes more flat while the temperature at the interface increases. Finally, at steady-state, the same temperature is found everywhere along the radial direction.

Figure 3.2: Temperature profiles at axial(top) direction and at radial direction(bottom) at two distinct axial coordinates x before steady state

3.1.2 *When the insulator isn't there*

So much for the conduction, Let's now imagine that the insulator is removed and the thermal conductor has a direct contact with ambient environment. Compared with the conduction experiment, it is obvious that we are dealing with a different boundary condition. Naively, you might think that the condition of $q = 0$ no longer hold at the conductor-air interface. However, it does hold, but for a different reason. To see this, let's first recall the Seebeck effect [1] where heat is carried by electron flow from hot end to cold end (see the bottom inlet in Fig.3.3). At the solid-air interface,
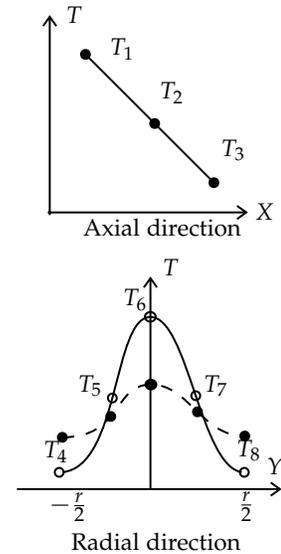
[1] Thermoelectric effect on Wikipedia

the electron that carries thermal energy is bounded to positively charged nuclear in the solid. It is unlikely for free molecules in the air to carry electrons out of solid body. Thus, **the flux of heat flow defined in Fourier's law must be zero at the solid-air interface because the electrons that carry the thermal energy cannot jump off the solid surface by themselves** (see the top inlet in Fig.3.3). As a result, the boundary condition of $q = 0$ remains unchanged. But $q = 0$ **does not mean no heat exchange at the interface**. We know this because we use heating radiator to warm our room up in the winter. Obviously, Fourier's law describes energy flow by conduction, and is no longer sufficient in this scenario. We need a different law to describe heat transfer process at the interface. It was Sir Isaac Newton who first found that interfacial heat transfer rate $\dot{Q}_{conv}$, is proportional to the temperature difference between the solid surface T_s and the air T_∞. In mathematical terms, this law can be written as

$$\dot{Q}_{conv} = h_m A_s (T_s - T_\infty) \tag{3.1}$$

with h_m and A_s being "**convection coefficient**" and surface area, respectively. With Eqn. (3.1), we are now well-equipped to derive another governing equation for the convection and conduction process in a solid by employing the conservation law of thermal energy.

Similar to what we did in Lab1, we cut a thin slice out of an one-dimensional thermal conductor with arbitrary geometry(see Fig.3.4). Here we again have some heat flux q_l coming in at the left and some coming out at the right (q_r). On the surface, molecules in the air take heat away with a rate of $\dot{Q}_{conv}$. The conservation of energy tells us that the total change in the thermal energy of the slice ΔQ is simply

$$\Delta Q = q_l A_{c,l} dt - q_r A_{c,r} dt - \dot{Q}_{conv} dt, \tag{3.2}$$

where $A_{c,l}$ and $A_{c,r}$ are the cross section areas at left and right faces of the slice, respectively. Recall that, from Lab2, the change in ther-
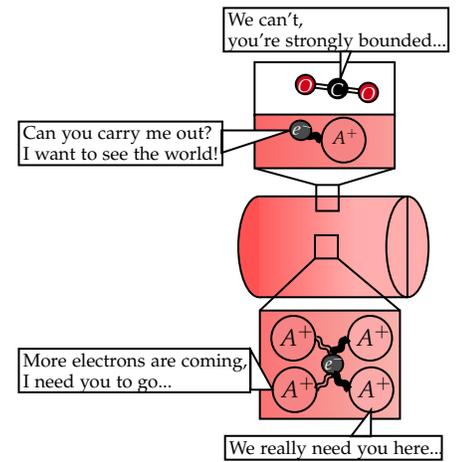

Figure 3.3: Electrons' behavior in conductor body(bottom inlet) and solid-air interface(top inlet). The wavy lines represent interactions between different particles, and the weight of those lines indicates the interaction strength.

mal energy ΔQ is related to **the change in temperature** $T_l - T_r$ by the following equation:

$$\Delta Q = mc_p(T_l - T_r) \tag{3.3}$$

where the mass of the slice is $m = \rho \int_0^{\delta x} A_c(x)dx$ with δx being slice thickness. We now write $\dot{Q}_{conv}$ explicitly using Eqn.(3.1), and divide both sides of Eqn. (3.2) by dt to give

Figure 3.4: A slice of conductor with non-uniform cross section

$$\rho \int_0^\delta A_c(x)dx \frac{\partial T}{\partial t} = q_l A_{c,l} - q_r A_{c,r} - h_m A_s(T - T_\infty). \tag{3.4}$$

Eqn.3.4 implies two assumptions: *(1) the change in temperature is one-dimensional so the surface temperature is equal to the body temperature, and (2) the change in temperature is negligible at the slice outer surface(i.e. T is not a function of x at surface).* With these assumptions, we now take a first derivative with respect to x at both sides of Eqn.3.4 to give

$$\rho A_c(x) \frac{\partial T}{\partial t} = -\frac{\partial(q A_c)}{\partial x} - h_m \frac{dA_s}{dx}(T - T_\infty). \tag{3.5}$$

Once again, we can derive a governing equation free of flux q by applying **Fourier's law** to give

$$\rho A_c(x) \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left(-\kappa \frac{\partial T}{\partial x} A_c\right) - h_m \frac{dA_s}{dx}(T - T_\infty). \tag{3.6}$$

Eqn.3.6 is the so-called *heat equation for extended surface*.

Newton's cooling law gives the macroscopic description of convection. At atomic level, the molecules in the air collide with the hot surface while taking some energy off the solid through the process of elastic collision [2]. But it was not long after John Tyndall [3] showed his fellow scientists how to measure infrared emission from a platinum filament, people started to wondering whether Eqn.(3.1) is still valid as there is one more mechanism for interfacial heat transfer except convection, that is, thermal radiation.
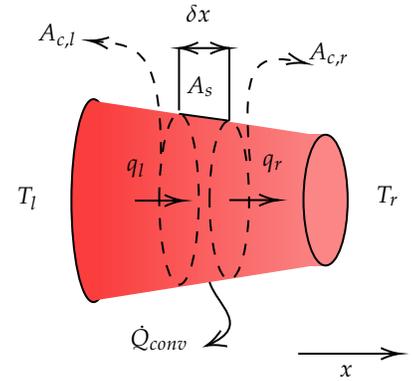
[2] Elastic collision on Wikipedia

[3] John Tyndall on Wikipedia

3.1.3   *Buoyancy: things happen to heated molecules*

Before we discuss radiative heat transfer, there are something about the heated air worth mentioning. Why do we care about what happen to them? The answer to this question helps us to understand the concept of **buoyancy**, which is a key fluid property to engineers when they want to design a proper cooling system for large infrastructures such as internet server farm. After one or multiple collisions, ambient molecules take thermal energy off the solid surface, which adds to their kinetic energies. Those hot molecules have a tendency of fast moving in the ambient environment, and travel longer distances than those cold molecules. **At macroscopic level, a large group of hot gas molecules expands in the ambient environment, resulting in decreased density.** Now imagine that the hot air is contained in a mass-less but thermally insulating balloon which allows the hot air to expand freely. [4] According to Archimedes' principle, the cold air applies an upward force to the balloon, F_b, and

$$F_b = -\rho_{cold}gV \tag{3.7}$$

with ρ_{cold} and V being the density of cold air and the balloon volume in our case. Because the density of hot air, ρ_{hot}, is less than ρ_{cold}, F_b is larger than the gravity of hot air. Without any influence of wind, the mass-less balloon should float upwards until it reaches an altitude where $\rho_{cold} = \rho_{hot}$. [5]

Knowing the buoyancy of hot air can help us to see why natural convection is not an efficient way to dissipate heat in large-scale infrastructure. From Eqn. (3.1) the convective heat transfer rate depends on temperature difference between the heated body and its surrounding. However, the heated air near the bottom of a large body will float upwards to the top where the temperature difference is no longer large because of the accumulation of hot air. As a result, the upper part of the body will be overheating. For a large-

[4] In the real world, the pressure inside a balloon is always slightly higher than the pressure outside of the balloon. This phonomenon is described by the **Young–Laplace equation**

[5] In the reality, hot air balloons are operating upon the similar principle. A fun fact about commercial balloons is that one cubic meter air is able to drag a mass of 7grams when it is heated to $100^0 F$.

scale electronic infrastructure, overheating is the primary killer of device health. To avoid this problem, engineers have designed various forced cooling methods, such as water cooling and fan cooling.

3.2   A Few Words on Radiation

Radiation is perhaps the most special one among the three primary heat transfer mechanisms as it relies on neither solid nor fluid to transport heat. Instead, it is the electromagnetic wave that carries thermal energy and gets emitted or absorbed at surface of solids. People postulate the effects of radiation of visible light since ancient farming society without knowing its electromagnetic nature. Then, in 1888, German physicist **Heinrich Hertz** succeeded in demonstrating the existence of long-wavelength electromagnetic waves and showed that their properties are consistent with those of the shorter-wavelength visible light [6]. Two decades before that, **James Maxwell**, in his formalation of electromagnetism, describes light as a propagating wave of electromagnetic fields, from which he predicted the existence of electromagnetic radiation.

In the case of heat transfer by thermal radiation, *we can write radiative heat flux q_{rad} using a form similar to Newton's cooling law(see Eq.(3.8)).* People rely on this formalism because the mean heat transfer coefficient h_m can then be written as a sum of convective $h_{m,c}$ and radiative coefficients $h_{m,r}$(see Eq.(3.9)).

$$q_{rad} = h_{m,r}(T_s - T_\infty) \qquad (3.8)$$

$$q = q_{conv} + q_{rad} = (h_{m,c} + h_{m,r})(T_s - T_\infty) = h_m(T_s - T_\infty) \qquad (3.9)$$

As a result, Eqn.(3.6) is still the governing equation for convective+radiative heat transfer process at the heated surface. To get an explicit expression for $h_{m,r}$, we resort to the Stefan-Boltzmann law in which the heat transfer rate varies as the difference in the 4th
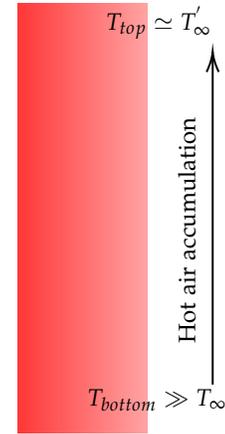

Figure 3.5: Hot air accumulation due to the buoyancy

[6] Light as electromagnetic radiation on Britannica

powers of temperature of solid surface and of ambient environment. In mathematical terms, the radiative heat flux q_{rad} is

$$q_{rad} = \xi \sigma F \left(T_s^4 - T_\infty^4 \right) \qquad (3.10)$$

where ξ is the emissivity of the object, F is the so-called "view factor", and Stefan-Boltzmann constant $\sigma = 5.6703 \times 10^{-8} watt/\text{m}^2 \text{ K}^4$ [7]. We expand the RHS of Eq.(3.10) to give

$$q_{rad} = \xi \sigma F(T_s + T_\infty)(T_s^2 + T_\infty^2)(T_s - T_\infty). \qquad (3.11)$$

From Eq.(3.11), the "*radiative heat transer coefficient*" is then

$$h_{m,r} = \xi \sigma F(T_s + T_\infty)(T_s^2 + T_\infty^2). \qquad (3.12)$$

By taking a closer look at Eq.(3.12), we notice that $h_{m,r}$ heavily depents on T_s and T_∞, and can not be treated as a constant if significant variation of those temperatures exist in the system. Based on this, people argued that Eq.(3.8) is only valid when the temperature on solid surface is moderately different from ambient temperature, and the surface should be well separated from other surfaces in the system [8].

3.3   *Juice from the History*

According to Nikola Tesla, energy, frequency, and vibration tell the secrets of our universe. However, the concept of energy was not well-understood by the scientific community untill 19th century. Before that, age-defining genius such as Isaac Newton, was even afraid of admitting that "temperature" and "heat" are two different things. In his anonymously published paper, **"Scala graduum Caloris. Calorum Descriptiones & signa"**, Newton used a small font size at the end to give the first description of the so-called **Newton's cooling law** [9], where he argued that the amount of heat getting off solid surface is due to the difference in "**temperature**",

[7] $\xi = 1$ for black body and $\xi = 0.95$ for the coated surface in our experiment, and the view factor $F = 1$.

[8] Researchers have found that the evanescent waves generated by the reflection of electromagnetic waves inside matters also promote the rate of heat transfer between two surfaces separated only by a small gap, see Fig. 1 in this 2018 paper here

[9] Newton's law is recently verified by two Japanese researchers using IR camera, and they found Newton's results were "quite accurate".(https://doi.org/10.1016/j.ijheatmasstransfer.2020.120544))

not "**heat**", between the surface and the environment. Standing at the frontline of mordern physical science, the situation that we're facing now is almost identical to what Newton encountered during his time. Mordern-time physcists argue that our classical world is essentially quantum-mechanical and stochastic at atomistic level. So they believe there must exist an ultimate unification between quantum mechanics and general relativity, with the later describing phenomena at length scale of galaxies. Will scientists find that quantum mechanics and general relativity might be indeed two distinct things just like how we distinguish "heat" from "temperature"? Only the future can tell. For now, let's just worry about things from Newton's time.

Lab4 Convection at Surface of Lumped Capacitance

In Lab4 we are going to explore the efficiency of convection on the surface of different solids as we change airflow rate and sample orientations. The principal theories behind this lab have been elaborated in previous notes, and we will review what we have learned so far along the way.

4.1 How to be Lumped?

In a system that transfers heat, we call some parts in the system as "lumped components" whenever their *internal temperature variation is negligible*(i.e., the distribution of temperature is almost uniform). Because of this, we can treat the lumped components as simple thermal resistance/capacitance, which simplifies the analysis of heat transfer process. But how do we know if internal temperature distribution in a subject is uniform enough?

First we notice that the relative size of lumped components within a heat transfer system matters as we might not be able to ignore what's going on inside a component of a size similar to the whole system. In other words, we need the *characteristic size* of lumped components to be small comparing to the whole system. Second, we learned from our previous experiments that thermal conductivity coefficient affects steepness of temperature profile in medium that transport heat. To see how the thermal conductivity of different materials affects temperature distribution inside material body, let's first imagine a metal rod wrapped in insulating

material, and with temperature at its two ends fixed at T_{end}. At the middle point, there is a heat source that generates constant heat flux ,q_{mid}(see the top schematic in Fig. 4.1).

At the steady state, we ignore temperature variation at radial direction, and obtain a mountain-like temperature profile along the axial direction (see the bottom profiles in Fig.4.1). With the constant q_{mid}, the "steepness" of temperature profile decreases as the thermal conductivity coefficient κ increases according to Fourier's law. As a result, the "height" of our mountain will decrease if the rod is made of materials that have larger thermal conductivity coefficient. We thus conclude that to make temperature variation inside heated body negligible, **the body needs to be highly thermal-conductive.**
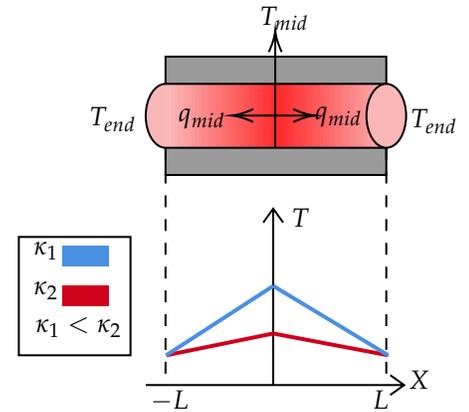

Figure 4.1: A metal rod with fixed end temperature(top), and its steady-state temperature profiles along axial direction(bottom) when it's made of two distinct materials with thermal conductivity coefficient being κ_1 and κ_2, respectively.

4.1.1   *Biot number*

*Being small and thermal-conductive is **NOT ENOUGH** for a heated body to be treated as lumped resistance or capacitance*, because so far, we deliberately overlooked the effects of convection at the two ends of the rod in the thought experiment above. Now, if we allow T_{end} to vary by exposing the two ends to an ambient environment. If the ambient temperature T_∞ is fixed, then the convective flux at the two ends are $q_{conv} = h(T_{end} - T_\infty)$, where h is convective heat transfer coefficient. Let L be the distance from the middle point to the rod end. Then, at the steady state, we establish conservation of energy in left(and right) half of the rod as

$$q_{mid} = q_{conv}$$
$$\kappa \frac{T_{mid} - T_{end}}{L} = h(T_{end} - T_\infty) \tag{4.1}$$

where T_{mid} is the middle-point temperature, and Fourier's Law and Newton's cooling law are used at the second equivalence. If all the temperatures in Eq.(4.1) are moved to LHS, we get a dimensionless number at RHS, which is named after the French physicist **Jean-Baptiste Biot** [1], i.e.,

[1] Biot on Wikipedia

$$\frac{T_{mid} - T_{end}}{T_{end} - T_\infty} = \frac{hL}{\kappa}. \tag{4.2}$$

With the relation in Eq.(4.2), we are now ready to use **Biot number** to judge whether a heated body can be treated as a lumped component. Because we require lumped components to have only negligible internal temperature gradient, $T_{mid} - T_{end}$ must be small (i.e. $T_{mid} - T_{end} \approx 0$). Since $T_{end} > T_\infty$ when the body is heated up, **the smaller the Biot number is, the more accurate the assumption of lumped component becomes.** A rule of thumb is that we can treat a body as a lumped capacitance/resistance, if its Biot number $Bi = \frac{hL}{\kappa} < 0.1$.

4.1.2   *Nussalt number and thermal conductivity of fluid*

It is not uncommon that people mistake Nussalt number with Biot number, because they have identical expression at first sight. While Biot number can be heuristically understood as a ratio of $\frac{\text{surface convection}}{\text{conduction in solid}}$, Nussalt number describes heat transfer solely inside the fluid body. Therefore, the thermal conductivity coefficient in Eq.(4.3) is a property of the fluid in contact with a solid surface.

$$Nu = \frac{hL}{k_{fluid}}. \tag{4.3}$$

A simple derivation of Nussalt number is given as following: Let us imagine a heated solid surface is in contact with a fluid of thickness L. When the fluid flows steadily along the surface, the heat flux is then

$$q_{conv,fluid} = h(T_s - T_{fluid}), \tag{4.4}$$

where T_s and T_{fluid} are the surface temperature and the temperature of fluid body. If there is no flow, on the other hand, the heat transfers to the fluid through heat conduction only to give

$$q_{cond,fluid} = k_{fluid} \frac{T_s - T_{fluid}}{L}. \tag{4.5}$$

The Nussalt number is then defined as the ratio of $\frac{\text{convection flux}}{\text{conduction flux}}$, i.e.,

$$Nu = \frac{h(T_s - T_{fluid})}{k_{fluid}\frac{(T_s - T_{fluid})}{L}} = \frac{hL}{k_{fluid}}. \tag{4.6}$$

For readers who enjoyed the discussion in Lab2, there is something off regarding the conduction in the fluid. In lab2 we have discussed how the heat flux represented by the flow of electrons is not well-defined at the **interface between the solid and the ambient environment**, and thus, $q_{conv} = 0$ is retained. Within the body of fluid, the conductive flux is, however, well-defined. Instead of having electron flow that carries thermal energy in solid, *the heat conduction in fluid is accomplished by the energy exchange between adjacent molecules.* For a polyatomic molecules [2], the energy exchange (or collisions) causes changes in vibrational and rotational frequencies, and translational velocities. The table below lists thermal conductivity of some common gases.

[2] In fact, it is possible to derive thermal conductivity coefficient from statistically averaged molecular properties, such as mean free path and mean molecular velocities. The methods used for such derivation is under the study of a subject called "statistical mechanics"

| | | Thermal conductivity coefficient in $mWm^{-1}K^{-1}$ | | | | | |
		100 K	200 K	300 K	400 K	500 K	600 K
	Air	9.5	18.5	26.4	33.5	39.9	46.0
Ar	Argon (P = 0)	6.3	12.4	17.7	22.4	26.5	30.3
BF_3	Boron trifluoride			19.0	24.6		
HCl	Hydrogen chloride		9.2	14.5	19.5	24.0	28.1
F_6S	Sulfur hexafluoride (P = 0)			13.0	20.6	27.5	33.8
H_2	Normal hydrogen (P = 0)	68.2	132.8	186.6	230.9	270.9	309.1
H_2O	Water (P=0)			18.6	26.1	35.6	46.2
D_2O	Deuterium oxide (P = 0)			18.2	26.6	36.3	47.6
H_2S	Hydrogen sulfide			14.6	20.5	26.4	32.4
H_3N	Ammonia			25.1	37.2	53.1	68.6
He	Helium (P = 0)	74.7	118.3	155.7	189.6	221.4	251.6
Kr	Krypton (P = 0)		6.5	9.5	12.3	14.8	17.1
NO	Nitric oxide		17.8	25.9	33.1	39.6	46.2
N_2	Nitrogen	9.4	18.3	26.0	32.8	39.0	44.8
N_2O	Nitrous oxide		9.8	17.4	26.0	34.1	41.8

Table 4.1: Tabulated data of thermal conductivity from NIST. Unless otherwise stated, the measurements are taken under 1 standard atmosphere pressure. The notation "P=0" indicates that the low-pressure limiting value is given.

From Table 4.1, we see that the thermal conductivity coefficient of gases increases with temperature. To explain such phenomenon, *it is not sufficient, however, to resort to ideal gas law*,

$$PV = nRT. \tag{4.7}$$

Because most of the data in Table 4.1 are obtained with the pressure P fixed at 1 bar, an increase in ambient temperature results in volumetric expansion, reducing the gas number density $\rho = n/V$. Since

reduced ρ indicates that gaseous molecule need to travel longer distance before its next collision, the ideal gas law seems to tell us that the rate of molecular energy exchange is quenched at high temperature, resulting decreasing thermal conductivity coefficient.

A heuristic correction to this contradictory explanation lies in the increased kinetic energy of gaseous molecules at high temperature. The excess of kinetic energy promotes the rate of collision despite of the increased distance between molecules, and hence, the rate of energy exchanges is also increased. For readers interested in a more quantitative relationship between temperature and thermal conductivity, a short account of the kinetic theory of monoatomic gas is prepared at the end of current chapter.

4.2   *Cooling and heating lumped components*

Now that we are persuaded that heated bodied with $Bi < 0.1$ are lumped component, it is easy to write down a governing equation, and the solution to it tells transient temperature variation of lumped body over time. Notice that the assumption of lumped component gives us the freedom of ignoring heat conduction inside heated body. So we can establish a conservation of energy by only considering the convection at solid surface, i.e., the thermal energy change in lumped component ΔQ is solely caused by the convective heat flux q_{conv} on the solid surface of area "A". From Eq.(2.3), we have

$$\Delta Q = mc_p(T_{init} - T_{body}) = q_{conv}Adt, \qquad (4.8)$$

where T_{init} and T_{body} are initial and current temperature in the lumped component body of interest [3]. Using Newton's cooling law at RHS of Eq.(4.8) gives

$$\frac{mc_p(T_{init} - T_{body})}{dt} = hA(T_{body} - T_\infty), \qquad (4.9)$$

where the surface temperature is replaced with T_{body} at RHS due to the assumption of lumped body. Now we divide both sides of (4.9)

[3] Remember that c_p is specific heat

by $T_{init} - T_{\infty}$ and write LHS as a first-order derivation with respect to time, which results in

$$-\frac{mc_p}{(T_{init} - T_{\infty})}\frac{dT_{body}}{dt} = hA\frac{T_{body} - T_{\infty}}{T_{init} - T_{\infty}}. \qquad (4.10)$$

Since $(T_{init} - T_{\infty})$ is a constant, Eq.(4.10) can be written as an ordinary differential equation (ODE) of the normalized temperature, $\theta = \frac{T_{body} - T_{\infty}}{T_{init} - T_{\infty}}$, as

$$-mc_p\frac{d\theta}{dt} = hA\theta. \qquad (4.11)$$

Eq.(4.11) has a simple solution of exponential function,i.e.,

$$\theta = exp\left(-\frac{hA}{mc_p}t\right) = exp(-\alpha t). \qquad (4.12)$$

Here the constant α (its inverse $\tau = 1/\alpha$ is sometimes referred as time constant) can be further simplified by writing the volume of lumped body, V, as a product of its surface area A exposed to fluid, and the inverse of its *surface-to-volume ratio* r_{sv} (i.e., $V = A/r_{sv}$). So Eq.(4.12) becomes

$$\theta = exp\left(-\frac{hr_{sv}}{\rho_s c_p}t\right) \qquad (4.13)$$

with ρ_s being the density of lumped component of interest. Eq.(4.13) sometimes is referred as power-law function of lumped component. It clearly shows that the transient temperature variation of lumped components *has nothing to do with the thermal conductivity coefficient.* Instead, the variation is controlled by (1) convective transfer coefficient,(2) surface-to-volume ratio, (3) solid density, and (4) specific heat. *Any action that reduces inverse time constant "α" will slow down the cooling process of lumped component.* Fig.4.2 shows that the reduction in T_{body} within first 4 units of time is the most significant when the constant α is the largest.

The discussion above can be readily applied to the heating process of lumped component. To see this, we write out temperatures
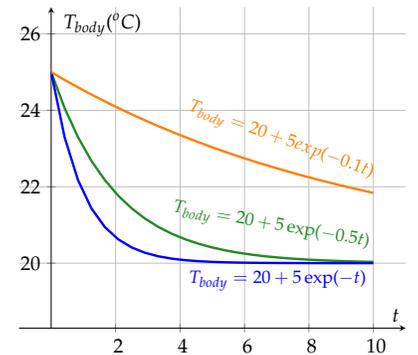

Figure 4.2: Temperature profiles as a function of time "t" with $T_{\infty} = 20^oC$, $T_{init} - T_{\infty} = 5^oC$, and time constant $\alpha = 1$(blue),0.5(green), and 0.1(orange).

explicitly in Eq.(4.13) as

$$T_{body} = T_\infty + (T_{init} - T_\infty) \, exp\left(-\frac{hr_{sv}}{\rho_s c_p}t\right).\qquad(4.14)$$

When the ambient temperature T_∞ is higher than the lumped body temperature T_{body}, Eq. (4.14) describes temperature variation of a heating process, and a cooling process otherwise. Fig.4.3 shows that the temperature profiles of a cooling and a heating process are mirror images to each other when $|T_{int} - T_\infty| = 3^oC$.
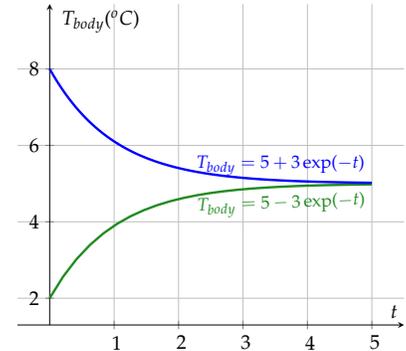

Figure 4.3: Cooling(bottom) and heating(top) process with $|T_{init} - T_\infty| = 3^oC$ and $T_\infty = 5^oC$

4.2.1 Effects of material properties on convection

Things become more interesting if we pay extra attention to the four factors in inverse time constant α, and their effects on the cooling/heating rate of lumped component. From Eq. (4.13), the density ρ_s and the specific heat c_p are two intrinsic material properties, and product of the two varies with material types. The table below lists the values of $\rho_s c_p$ for some common matallic and ceramic materials. If we fix the values of h and r_{sv}, Table 4.2 shows that steel and

Material	Specific heat(kJ/kgK)	Density(kg/m3)	$c_p \rho_s$
Aluminum	0.9	2550	2295
Brass	0.375	8730	3273.75
Stainless steel	0.49	8030	3934.7
Macor	0.79	2520	1990.8

Table 4.2: Specific heat and density of some common materials, the data for machinable glass ceramic "Macor" is obtained from Corning Inc.

Macor have the smallest and the largest inverse time constant α, respectively. Starting from the same initial temperature, the rank of cooling/heating rates for these materials follows a sequence of Macor>Aluminum>Brass>Stainless steel.

4.2.2 Effects of geometry of lumped components on convection

There is no way to fit a decent discussion of the relationship between convective coefficient h with other physical parameters in this note [4], which left us with the last factor in α: the surface-to-volume ratio, r_{sv}. For some highly symmetric shapes, such as sphere and

[4] Up to March 24th, 2021, there are in total 1,360,000 research papers related to "convective coefficient" available on Google scholar

cube, if we assume that the whole lumped body is exposed to convective process, then r_{sv} can be readily calculated by using formula listed in Table 4.3. With these formula, we can also plot r_{sv} against

Geometry	Vol.	r_{sv}
Tetrahedron	$\sqrt{2}a^3/12$	$14.697/a$
Octahedron	$\sqrt{2}a^3/3$	$7.348/a$
Cube	a^3	$6/a$
Sphere	$4\pi a^3/3$	$3/a$
Dodecahedron	$(15+7\sqrt{5})a^3/4$	$2.694/a$

Table 4.3: r_{sv} of common shapes with a being the length of edge/radius

the volumes of these shapes in Fig.4.4. This simple analysis shows that for a given volume of lumped component, tetrahedron has the largest r_{sv} while sphere has the smallest. Because large r_{sv} results in large α, Fig.4.2 and 4.4 indicate that *the T_{body} drops(cooling) and increases(heating) much quicker in a tetrahedral body than in a spherical body*. As the lumped body becomes more and more bulky, r_{sv} of all kinds of polyhedral bodies decrease in an order of $O(1/x)$, resulting in more sluggish cooling and heating process. Of course, the bulky component also undermines the validity of lumped capacitance assumption.
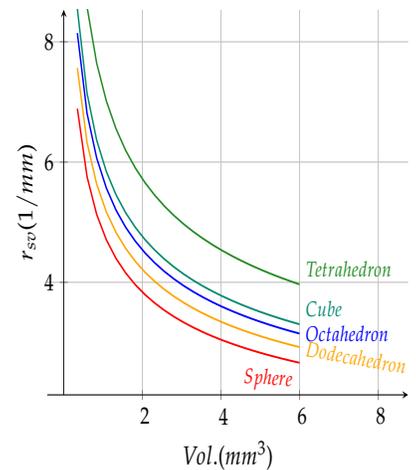
4.3 A kinetic theory for thermal conductivity of gas

```
TL;DR: I indulged myself to write this section only
because deriving macroscopic law from statistical
behavior of molecules is intellectually satisfying.
For impatient readers, please check the most important
results at the end.  For ambitious readers, please refer
to Sears' book for accessible introduction to statistical
thermodynamics 5, and C.V. Heer's account of stochastic
perspectives of thermal physics 6.
```

To simplify our derivation without losing generality, we again employ some of the assumptions introduced in Lab1, and they are

I. gas molecules are treated as hard spherical mass of radius r, and

II. potential energy of each molecule is negligible.

Figure 4.4: r_{sv} versus polyhedral volumes

[5] Francis Weston Sears, Gerhard L Salinger, and John E Lee. *Thermodynamics, kinetic theory, and statistical thermodynamics*. Addison-Wesley, 1975

[6] Clifford V Heer. *Statistical mechanics, kinetic theory, and stochastic processes*. Elsevier, 2012

Careful readers might find that assumption I is now considering molecular volumes comparing to the old one. In lab1, we only care about the collision between molecules and the wall of container where the cross section of single molecule is negligible. Inside the body of a container, where molecules collide with each other, making molecular size finite helps to visualize process of momentum/energy exchange and to calculate important physical quantities that describe molecular motions. One of such physical quantities is the so-called *mean free path*(MFP), which measures on average the distance a gas molecule needs to travel before it collides with another molecule. Knowing MFP is important for our purpose of relating thermal conductivity to molecular motions.As we will show in later sections, viscosity and conductivity stem from momentum and energy exchange between adjacent gas molecules, and MFT provides a measurement of how often such exchanges could happen and bridges thermal conductivity with other thermodynamic quantities (i.e. state parameters). So let's start by deriving a simple expression for MFP below.

4.3.1 *Scattering and Mean free path*

From the assumption I, we can define the exclusion volume for single molecule as $4\pi r^3/3$, i.e., two gas molecules can not have overlap in their molecular volumes. Let us now consider a thin slice of thickness Δx in a cubic container of edge length L. The slice is thin enough so that molecules contained in the slice does not overlap with each other long $x-$dimension (see Fig. 4.5).

Let n be the number of gas molecules per unit volume in the container, and the number of molecules in the slice is then $n \times L^2 \times \Delta x$. Consider now that a beam of cross section area $L \times L$ is shooting towards the slice. Because of the assumption I, we can treat molecules in the slice (i.e. the "target") as hard spheres of diameter $4r$, and treat molecules in the beam (i.e., the "bullet") as point mass to enforce the assumption of finite molecular size. Bul-
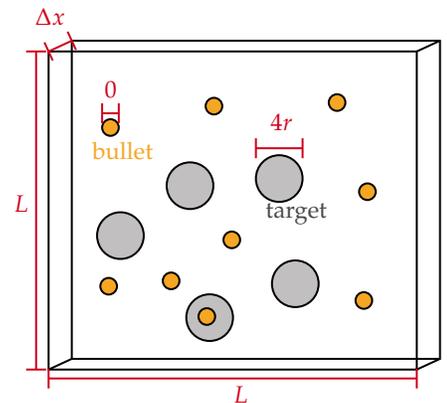

Figure 4.5: Scattering cross section area due to molecules in a $L \times L \times \Delta x$ slice.

lets that hit on targets will be scattered out of the beam, and the area that can be hit by bullets is called the *scattering cross section*. In our case, the cross section area is simply $4\pi r^2 \times n \times L^2 \times \Delta x$. Let N and ΔN be the total number of bullets and **change in bullet number after the beam's pass through the slice**. Then we have the following relation,

$$\frac{\Delta N}{N} = \frac{\text{scattering cross section area}}{\text{total cross section area}} = \frac{-4\pi r^2 \times n \times \cancel{L^2} \times \Delta x}{\cancel{L^2}}, \quad (4.15)$$

where the minus indicates decrease in total number of "bullets". Integrating both sides of Eq.(4.15) gives an expression for the number of "bullets" that survive scattering at x as

$$N = N_0 \exp\left(-4\pi r^2 n x\right) \quad (4.16)$$

with N_0 being total number of "bullets" at $x = 0$. If we define $\sigma = 4\pi r^2$ as cross section area of single molecule and substitute Eq.(4.16) back into (4.15), we obtain an expression for **"bullets" number that collide with "targets" at arbitrary coordinate** x as

$$\Delta N(x) = \sigma n \Delta x N_0 \exp(-\sigma n x). \quad (4.17)$$

Using Eq.(4.17) we can calculate the *averaged travel distance before any one of N_0 bullets collides with a "target"* as

$$l = \frac{\Sigma x \Delta N}{N_0} = \sigma n \int_0^\infty x \exp(-\sigma n x) dx = \frac{1}{\sigma n}. \quad (4.18)$$

Eq.(4.18) gives an expression for mean free path, and it can be further modified by noticing that the "target" molecules are treated as still objects. In reality, the "targets" are also moving, but here we will stick to this simpler form of MFP to proceed our analysis [7]. By reversing our logic above, we can conclude that MFP from molecule's last collision is also given by Eq.(4.18), i.e., *at any given time instance, gas molecules on average have traveled l from their last collisions and will need to travel another l to have another collision*. We will use this conclusion below to drive viscosity η and thermal conductivity κ of gases.

[7] On the assumption that all molecules have the same speed, Clausius obtained the result $l = \frac{0.75}{\sigma n}$.

4.3.2   *Viscosity from momentum transport*

If we substitute σ and n for oxygen molecules in Eq.(4.18), MFP between two subsequent collisions of single oxygen molecule is about $10^{-7}m$, three orders of magnitude larger than the size of oxygen molecule. So it is really weird to find that nearly all the real gases are viscous since how come frictional forces exist among gas molecules at first place when they are far separated? To answer this question, let us imagine two parallel planes with one of them moving with a constant velocity of u to the right (see Fig. 4.6). The velocity u is much slower than molecular velocities, so the whole system can still be treated as if it's at equilibrium state. As a result, a linear velocity profile is found between the planes.

At an imaginary plane (e.g. the dashed line in the figure), there must be exchange of molecular momenta as molecules that pass the plane from below have on average slower velocities at $x-$direction than those pass it from above. ***Would it be that the viscosity of gas is resulted from such momentum exchange?*** To calculate the net momentum exchange at the imaginary plane, we first need to know **the average vertical distance $\bar{y}$ from the imaginary plane where molecules had last collision before they pass the plane**. As we will see later, $\bar{y}$ helps us to find the average $x-$momentum of molecules that pass the imaginary plane.

Based on the terminology we developed in Lab1, the mass flux of molecules arriving at the imaginary plane in a polar angle θ is obtained by integrating over azimuthal angle and velocity magnitude in Eq.(1.8), i.e,

$$\Phi_\theta = \frac{1}{2} n \bar{v} \sin(\theta) \cos(\theta) \Delta\theta. \tag{4.19}$$

The total mass flux is then

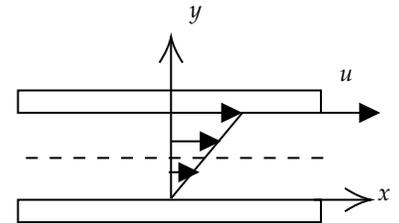$$\Phi = \int_0^{\pi/2} \Phi_\theta d\theta = \frac{1}{4} n \bar{v}. \tag{4.20}$$

Figure 4.6: Vertical velocity profile developed in gas between two parallel planes. The top plane moves with velocity u to the right

Given Eq.(4.19) and (4.20), the averaged vertical distance $\bar{y}$ is then

$$\bar{y} = \frac{n\bar{v}\int_0^{\pi/2}\sin(\theta)\cos(\theta)l\cos(\theta)d\theta}{2\Phi} = \frac{2}{3}l, \tag{4.21}$$

where $l\cos(\theta)$ is the vertical distance between the imaginary plane and the latest collisions that cause molecules arriving at the plane at a polar angle of θ (see Fig.4.7).

Let u_0 be the $x-$velocity at the imaginary plane, then the $x-$momentum of molecules pass the plane from above is them

$$p_x^+ = m(u_0 + \frac{2}{3}l\frac{du}{dy}), \tag{4.22}$$

according to Taylor expansion. Similarly, for those below the plane, their $x-$momentum is simply

$$p_x^- = m(u_0 - \frac{2}{3}l\frac{du}{dy}), \tag{4.23}$$

and the change in $x-$momentum from the "fast" to the "slow" is

$$\Delta p_x = m\frac{4}{3}l\frac{du}{dy}. \tag{4.24}$$

Multiplying both sides of Eq.(4.24) by the Φ in Eq.(4.20) gives the "viscous force" per unit area at the LHS [8], i.e.,

$$\frac{F_{vis}}{A} = \frac{4}{3}ml\Phi\frac{du}{dy}. \tag{4.25}$$

Since the viscosity coefficient η is defined through the equation

$$\frac{F_{vis}}{A} = \eta\frac{du}{dy}, \tag{4.26}$$

we finally obtain an explicit expression for η as

$$\eta = \frac{m\bar{v}}{3\sigma}. \tag{4.27}$$

Eq.(4.27) is very interesting as it indicates that the viscosity of gas does not depend on gas density but average molecular velocity. With the method developed here, deriving an expression for thermal conductivity coefficient κ becomes fairly easy.
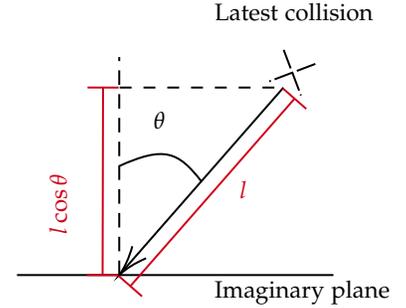

Figure 4.7: Vertical distance between the imaginary plane and the latest collision

[8] we used this trick before in Lab1

4.3.3   *Thermal conductivity from energy transport*

We first notice that Fourier's law looks a lot like Eq.(4.26). Instead of having viscous force per unit area at LHS, we have, in Fourier's law, the heat flux q proportional to temperature gradient, i.e.,

$$q = -\kappa \frac{dT}{dy}. \tag{4.28}$$

We again imagine an imaginary plane in the gas, at which the temperature is T_0. The energy of single molecule at the plane is $c'_v T_0$ with c'_v being the "specific heat" per molecule. By following the formalism in Eq. (4.22) and (4.23), the molecular energies above and below the plane before their arrival are

$$E_{above} = c'_v(T_0 + \frac{2}{3}l\frac{dT}{dy}) \tag{4.29}$$

and

$$E_{below} = c'_v(T_0 - \frac{2}{3}l\frac{dT}{dy}). \tag{4.30}$$

Following the same logic laid out in the previous subsection [9], we have

$$\Phi \times (E_{below} - E_{above}) = -\frac{c'_v \bar{v}}{3\sigma}\frac{dT}{dy} = -\kappa\frac{dT}{dy}, \tag{4.31}$$

and

$$\kappa = \frac{c'_v \bar{v}}{3\sigma}. \tag{4.32}$$

[9] Notice that dT/dy is negative here, so $E_{below} - E_{above} > 0$

From Eq.(4.32), we find that the thermal conductivity of gas is inversely proportional to the size of molecule. The gas molecule that can move fast and store lots of thermal energy tends to have better thermal conductivity.

4.4   *Inadequacy of hard-sphere assumption*

Based on our derivation of η and κ, the ratio of the two gives

$$\kappa/\eta = \frac{c'_v}{m} = \frac{c_v}{M} \tag{4.33}$$

where c_v and M are the specific heat and the molar mass of gas molecule, respectively. From this, we have

$$\frac{\eta c_v}{\kappa M} = 1. \tag{4.34}$$

Everything in Eq.(4.34) is experimentally measurable, so we can test this relationship on real gases. However, real gases usually have such value around 0.5, not 1 [10], which means our hard-sphere assumption does not capture the whole story of molecular collisions. But at least it got the order of magnitude right. To make the assumption more realistic, we might have no choice but to consider what is happening at the moment of molecular collision, an interesting realm for chemists, physicists, and me.

[10] For air, the ratio has a value of 0.51546.

Lab5 *Understanding Heat Exchanger*

```
"Of course, they are not directly called heat exchangers
in our daily life.  They can be found in many devices
around us:  computers, refrigerators, radiators, mobiles
phones and etc.  I promise many of us live with them every
day.  Just imagine a hot summer day, you have to sit a room
without an air conditioner or have a warm coke if you don't
have a fridge."---A comment from Dr.Qian
```

Unlike previous notes, in Lab5 we will focus on indicators of heat exchange efficiency and make quantitative comparison of various heat exchangers, i.e., engineering side of things. At the end, we will also discuss elementary design principles of heat exchanger.

5.1 *Basic Working Principles of Heat Exchanger*

Heat exchanger, by definition, is any device that exchange heat between two or more fluids [1] of different temperatures. The basic components of heat exchangers include: (1) flow path of cold fluid, (2) flow path of hot fluid, and (3) a separator to prevent direct contact between the cold and the hot [2].

Regardless of the design, the underlying principles that heat exchangers work upon are laws of thermodynamics [3]. While the zeroth law of thermodynamics demands us to *create temperature difference to simply make energy flow happen*, the first and second laws of thermodynamics are the guidelines for engineers to fine-tune the energy flow in heat exchangers to improve the energy efficiency while keep the operating cost modest.

The first law of thermodynamics dictates how we construct governing equation for describing energy flow in the exchangers. Be-

[1] Heat exchanger can also transfer heat between gas and liquid, a multi-phase process.

[2] some heat exchangers might allow direct mixing between two distinct fluids, so fluids are not totally separated in those exchangers

[3] Even though they are called the "laws", thermodynamics is actually an empirical subject, i.e., these laws are summarized solely based upon experimental results, and are not, at least for now, logically derivable.

cause the energy cannot be destroyed or created, any amount of energy goes into the heat exchanger will cause equal amount of decrease in energy in its surrounding environment. The governing equation is then established upon the conservation of energy, and has terms for what is happening inside heat exchanger at one side and has terms related to environmental changes the other side, i.e.,

$$\Delta U_{exchanger} = -\Delta U_{environment}. \tag{5.1}$$

The second law of thermodynamics sheds light on the way of how we are supposed to delay or accelerate the process of exchangers achieving their thermal equilibrium. A thermal equilibrium is achieved when the entropy [4] of the system is maximized. According to the second law, the only changes possible at an equilibrium state are the ones that further increase system entropy.

[4] the entropy is defined as the ratio of change in heat over temperature, i.e., $\Delta S = \frac{\Delta Q}{T}$

5.2 Common Types of Heat Exchangers

Heat exchangers fall into different classes based on their *flow configuration, construction method, and heat transfer mechanism*. Commonly used flow configurations are: concurrent flow, countercurrent flow, cross flow, and hybrid flow. The schematic of how the distinct fluid streams flow in these configurations are shown in Fig. 5.1 and 5.2, where no direct mixing is allowed in all the configurations. These flow configurations are usually adopted in three

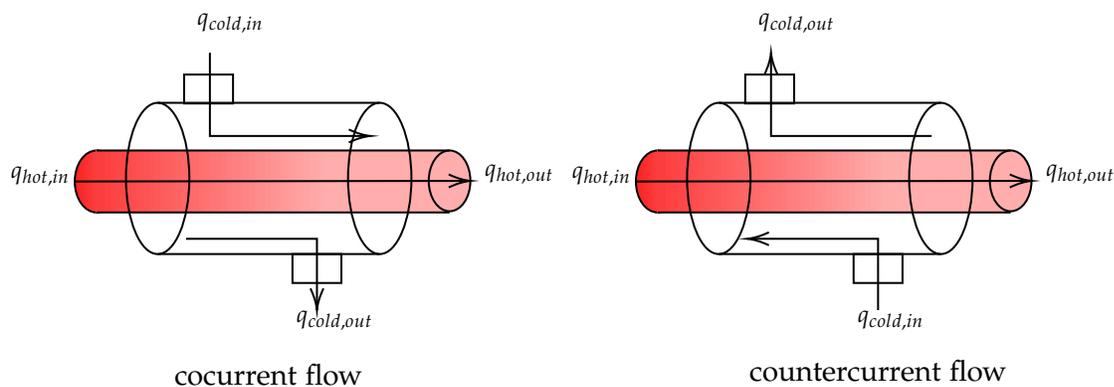cocurrent flow   countercurrent flow

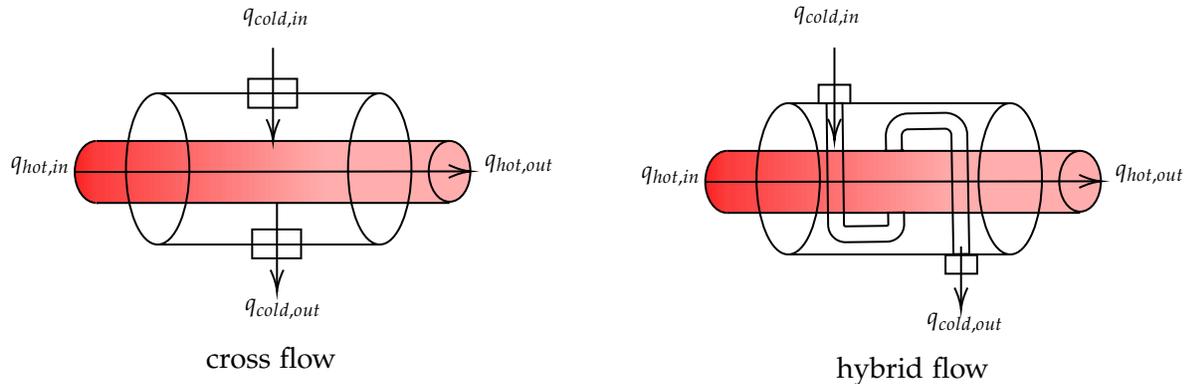Figure 5.1: Cocurrent and countercurrent flow configurations in heat exchangers

Figure 5.2: Cross and hybrid flow configurations in heat exchangers

types of exchangers,i.e., *"shell-and-tube", brazed plate, and concentric tube.* While the flow configurations sketched in the figures above can be directly applied to concentric tube exchanger, they might be modified in the later two exchangers because of their special arrangements of flow channels. In shell-and-tube exchanger, a large array of small pipes are regular arranged by keeping pipe interval constant. The array is then wrapped inside a large tube with fluid filling the space between the pipes and the tube. When it is operating, the fluid flowing in small pipes gets heated up or cooled down by the fluid flowing in the tube. The brazed plate exchanger, on the other hand, is constructed by bundling multiple corrugated thin plates together. Each pair of plates is welded and then stacked together to complete the flow channels. Unlike shell-and-tube and concentric tube exchangers, plate exchangers can allow more than two kinds of fluids flowing in separate channels to improve heat transfer efficiency.

The construction method of heat exchanger can be roughly classified using the following three criteria: *(1) recuperative vs. regenerative, (2) direct vs. indirect, and (3)static vs. dynamic.* In recuperative exchangers, fluids are flowing simultaneously in their own channels while regenerative exchangers allow hot/cold fluids flow in different channels alternatively. Among the recuperative exchangers, those allow direct mixing of different fluids are called direct exchangers, and they are indirect exchangers otherwise. Meanwhile,

the static-dynamic criterion only applies to regenerative exchangers. In static regenerator, the heat exchanger components remain stationary, while dynamic regenerator has its components moving during the heat exchange process. Comparing to recuperative exchangers, using both static and dynamic exchangers incurs the risk of cross-contamination between distinct fluids.

Condenser, evaporators, and boilers are also considered as heat exchangers in principle. Heat transports in these systems through multi-phase heat transfer mechanism while exchangers introduced above generally transfer heat in single-phase flow. The hero in this class of heat exchangers is the microchannel coil, which is widely used in the air conditioning systems for the refrigerant evaporation and condensation [5]. One fun fact is that the designs of microchannel-based exchangers generally exclude the use of water. The reason for such design can be attributed to the high boiling temperature of water in standard ambient environment. You certainly don't want your fridge and air conditioner overheating!

[5] more details can be found here

5.3 *Number of Transfer Units (NTU) Method*

The general heat exchanger selection considerations are: (1)size limitations, (2)thermal output, (3)fluid type, and (4)costs. Industrial applications of commonly used heat exchangers are listed in Table.5.1 below. We are not going to extend our discussion of these elements here as each one of them worth a Ph.D. thesis. Instead, we will just focus on defining efficiency of heat exchanger using the number of transfer units (NTU) method.

Type of heat exchanger	Common applications
Shell and tube	Oil cooling/refining
	Steam generation
	Industrial paint system
Concentric tube	Heat transfer process that has spatial constraints
Brazed plate	Food processing
	Furnaces

Table 5.1: Industrial applications of common heat exchangers

NTU starts with finding the maximum heat flux q_{max} by multiplying minimum *hate capacity rate* C_{min} with the largest temperature

difference in an exchanger [6], that is,

$$q_{max} = C_{min}(T_{h,i} - T_{c,i}).$$
(5.2)

$T_{h,i}$ and $T_{c,i}$ in Eq. (5.2) are the temperatures at hot and cold fluid inlet, respectively. The heat capacity rate is found by

$$C_{min} = \min\{\dot{m}_c c_{p,c}, \dot{m}_h c_{p,h}\},$$
(5.3)

and of course

$$C_{max} = \max\{\dot{m}_c c_{p,c}, \dot{m}_h c_{p,h}\}.$$
(5.4)

$\dot{m}_{c(h)}$ is the mass flow rate of cold(hot) fluids, and $c_{p,c(h)}$ is the specific heat of cold(hot) fluids. The net heat flux can be defined by using hot fluid temperatures at inlet and outlet as

$$q_{net} = \dot{m}_h c_{p,h}(T_{h,i} - T_{h,o}).$$
(5.5)

Then the effectiveness of heat exchanger ϵ is given as

$$\epsilon = \frac{q_{net}}{q_{max}}.$$
(5.6)

A simple dimensional analysis tells us that ϵ is a function of the heat capacity ratio $C_r = C_{min}/C_{max}$, and number of transfer units (NTU). NTU is given by

$$NTU = \frac{UA}{C_{min}}$$
(5.7)

where U and A are the overall heat transfer coefficient and the heat transfer area respectively. The product UA can be calculated using log-mean temperature difference ΔT_{lm} and q_{net} by

$$UA = \frac{q_{net}}{\Delta T_{lm}}.$$
(5.8)

The log-mean temperature difference(LMTD) for parallel flow(PF) and counter flow(CF) configurations are given by

$$\Delta T_{lm,PF} = \frac{(T_{h,i} - T_{c,i}) - (T_{h,o} - T_{c,o})}{\ln \frac{T_{h,i} - T_{c,i}}{T_{h,o} - T_{c,o}}}$$
(5.9)

and

$$\Delta T_{lm,CF} = \frac{\left(T_{h,i} - T_{c,o}\right) - \left(T_{h,o} - T_{c,i}\right)}{\ln \frac{T_{h,i} - T_{c,o}}{T_{h,o} - T_{c,i}}}. \tag{5.10}$$

The explicit relationship between ϵ and NTU for complicated geometries and flow configurations can only be obtained by solving a set of PDEs numerically. But for a parallel flow heat exchanger, the relationship is

$$\epsilon = \frac{1 - \exp\left[-NTU\left(1 + C_r\right)\right]}{1 + C_r}, \tag{5.11}$$

and for counter flow configuration we have instead [7]

$$\epsilon = \frac{1 - \exp\left[-NTU\left(1 - C_r\right)\right]}{1 - C_r \exp\left[-NTU\left(1 - C_r\right)\right]}. \tag{5.13}$$

[7] When $C_r = 1$, the effectiveness is

$$\epsilon = \frac{NTU}{1 + NTU} \tag{5.12}$$

It turns out to be a common practice *using LMTD to design the size of heat exchanger* [8]. Here we lay out the general procedures for finding the size of tubular heat exchangers from LMTD. From Eq.(5.8), the heat transfer area A is then

[8] Cüneyt Ezgi. Basic design methods of heat exchanger. In *Heat Exchangers-Design, Experiment and Simulation*. IntechOpen, 2017

$$A = \frac{q_{net}}{U\Delta T_{lm}}, \tag{5.14}$$

For the unfinned tubular heat exchangers, the overall heat transfer coefficient on outer surface is given by

$$U_o = \frac{1}{\frac{r_o}{r_i}\frac{1}{h_i} + \frac{r_o}{r_i}R_{fi} + \frac{r_o}{k}\ln\left(\frac{r_o}{r_i}\right) + R_{fo} + \frac{1}{h_o}}. \tag{5.15}$$

The subscript "o" and "i" in Eq.(5.15) represents outer and inner surfaces of heat exchangers, respectively. Using this terminology, r_o and r_i are radius of outer tube and inner tube while h_i and h_o are heat transfer coefficients of the fluid in inner tube, and the fluid flowing between inner and outer tubes. R_{fo} and R_{fi} are fouling resistance of the inside and outside surfaces, respectively. We consider fouling resistances here as it is well known that the fouling process of tube materials usually causes increase in thermal resistance. Substituting Eq.(5.15) back into Eq. (5.14) gives the area of outer tube, from which we can calculate the length of tube as $L_o = A/(2\pi r_o)$.

If the size and type of exchanger are known, we can use NTU to determine the outlet temperatures. To do so, we first find the heat capacity ratio C_r as

$$C_r = \frac{\min\{\dot{m}_c c_{p,c}, \dot{m}_h c_{p,h}\}}{\max\{\dot{m}_c c_{p,c}, \dot{m}_h c_{p,h}\}}. \tag{5.16}$$

Because the size of exchanger is known, we can then calculate the product of UA, and NTU using Eq.(5.7). Based on the flow configuration, we can determine the effectiveness ϵ using either Eq.(5.11) or Eq.(5.13). With calculated ϵ, we further use Eq.(5.6) to determine q_{net}, and $\Delta T_{lm} = q_{net}/(UA)$. As the final step, we use either Eq.(5.10) or (5.9) to calculate $T_{h,o}$ based on preset $T_{h,i}$, $T_{c,i}$ and $T_{c,o}$.

5.4   *Some Thoughts from a Newly-graduated Ph.D.*

It is not hard to understand heat exchanger, but designing an exchanger that fits engineering scenarios is always challenging. Let me conclude by quoting some of the questions raised regarding exchanger design in an email from a newly-graduated HVAC scientist. These are what engineers talk about when they talk about their passion for innovative exchanger design:

- What do we want to achieve (cooling the chips in the PC, freezing food in the fridge...)?

- How will this device be used in real-life (outdoor/indoor, mobile/stationary...)?

- Limitations and special considerations: Is there a space limitation like the AC system in the car? Or it has to be super reliable like coolers for the server/has high efficiency like new ACs to meet the strict standards...

- and finally, what about the cost (initial and operating...)? are there any trade-offs?

These interesting questions have demonstrated that the design and application of heat exchanger is not, as of yet, a closed book.

Bibliography

[1] Theodore L Bergman, Frank P Incropera, David P DeWitt, and Adrienne S Lavine. *Fundamentals of heat and mass transfer*. John Wiley & Sons, 2011.

[2] Daniel Bernoulli. Hydrodynamica. *Dulsecker. Consultable en ligne http://imgbase-scd-ulp. u-strasbg. fr/displayimage. php*, 1738.

[3] Stephen G Brush. History of the kinetic theory of gases. *Istituto della Enciclopedia Italiana*, 1, 2004.

[4] Cüneyt Ezgi. Basic design methods of heat exchanger. In *Heat Exchangers-Design, Experiment and Simulation*. IntechOpen, 2017.

[5] Richard P Feynman, Albert R Hibbs, and Daniel F Styer. *Quantum mechanics and path integrals*. Courier Corporation, 2010.

[6] Clifford V Heer. *Statistical mechanics, kinetic theory, and stochastic processes*. Elsevier, 2012.

[7] JC Maxwell. Illustrations of the dynamical theory of gases. *Philos Mag*, 19:19–32, 1867.

[8] Francis Weston Sears, Gerhard L Salinger, and John E Lee. *Thermodynamics, kinetic theory, and statistical thermodynamics*. Addison-Wesley, 1975.

Index